Analytic solutions of the 1D finite coupling delta function Bose gas

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An intensive study for both the weak coupling and strong coupling limits of the ground state properties of this classic system is presented. Detailed results for specific values of finite N are given and from them results for general N are determined. We focus on the density matrix and concomitantly its Fourier transform, the occupation numbers, along with the pair correlation function and concomitantly its Fourier transform, the structure factor. These are the signature quantities of the Bose gas. One specific result is that for weak coupling a rational polynomial structure holds despite the transcendental nature of the Bethe equations. All these new results are predicated on the Bethe ansatz and are built upon the seminal works of the past.

I. INTRODUCTION

The one dimensional delta function Bose gas is a classic in the field of exactly solvable integrable systems. Following on the realization that in the infinite coupling limit, the impenetrable limit, the system had many of the properties of the free Fermi gas [1], in their seminal work, Lieb and Liniger [2] solved the model exactly. They derived the Bethe ansatz and Bethe equations and went on to solve for the excitations in the thermodynamic limit [2, 3].

In this paper, we present an intensive study of the finite system, in the weak coupling and the strong coupling limits. Extensive analytical solutions are given for finite values of N, the particle number, and from them, the analytical solutions for general N are determined. Except for a couple of papers that give the excitations for a system of three particles [4] and larger number of particles [5, 6] this is the first intense study of this finite N body system for finite coupling. We concentrate on the principle quantities that are the signatures of the Bose gas.

After a preliminary Section II and Appendix A, introducing the Bethe ansatz and Bethe equations and their properties, we give extensive analytical solutions, in Sections III and IV, with the aide of Appendices B and C, for the density matrix and concomitantly the occupation numbers. In Sections V and VI, we do likewise for the pair correlation function and concomitantly the structure factor. As such we build upon the seminal work of Lenard [7], the very important work of Jimbo and Miwa [8], the Leningrad group [9], and our recent works [10, 11, 12]. We conclude the paper in Section VII, with some further comments following upon all these results.

The occupation numbers and structure factors for the modes are the experimentally realizable signatures of the Bose gas. The spur to the recent, over the past decade, revival of active interest in this system is due to Olshanii [13], who pointed out how this system could be realized in nature and to the current experimental activity seeking to realize it [14, 15, 16, 17, 18, 19]. We refer to the introduction in our earlier paper [10] for a discussion of the relevant physical parameters involved.

As all the new analytical solutions in this work are based upon the Bethe ansatz, it is appropriate to observe that this year celebrates the 75^{th} anniversary of Bethe's paper [20] in which he introduced the now famous ansatz in his study of the Heisenberg spin-chain. Ever since, it has been a golden key in unlocking the solutions to many exactly solvable integrable systems. Baxter [21] gave a concise review of these in his tribute to Yang, who has recently written [22] on the Bethe ansatz to honor Bethe.

II. CONSTRUCTING THE WAVEFUNCTION

The purpose of this section is to present the Bethe ansatz wavefunction and concomitantly the Bethe equations for the one-dimensional delta function Bose gas in periodic boundary conditions, as derived in the seminal paper by Lieb and Liniger [2]. The Schrödinger equation for this system with $\hbar = 1, 2m = 1$ is

$$\left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \le i < j \le N} \delta(x_i - x_j) \right) \psi_N(x_1, x_2, \ldots, x_N) = E \psi_N(x_1, x_2, \ldots, x_N) \tag{2.1}$$

where c controls the strength of the δ-function. Throughout this paper we consider only $c \ge 0$, the repulsive case.

Here we display the solution for periodic boundary conditions

$$\psi_N(x_1, x_2, \ldots, x_N) = \sum_{p \in S_N} a(p)\epsilon(p) e^{\left(\mathrm{i} \sum_{j=1}^{N} k_{p(j)} x_j\right)} \tag{2.2}$$

where S_N is the symmetric group on N symbols, and hence the wavefunction is a sum over $N!$ quantities. The function $\epsilon(p)$ is the signature of the permutation p, and the function $a(p)$ is given by

$$a(p) = \prod_{i<j}^{N} \left[1 + \frac{\mathrm{i}}{c}(k_{p(j)} - k_{p(i)})\right]. \tag{2.3}$$

Note that $a(p)$ may be given in a number of forms (any change absorbed into the normalisation), we choose (2.3) from Jimbo and Miwa [8]. Throughout this paper we use the normalisation given by

$$\mathcal{N}^2 = \int_{R_{N-1}} dx_1 \ldots dx_{N-1} |\psi_N(0, x_1, \ldots, x_{N-1})|^2 \tag{2.4}$$

where R_{N-1} is the domain of integration specified by

$$R_{N-1} : 0 < x_1 < \ldots < x_{N-1} < L. \tag{2.5}$$

In this paper we are concerned only with the ground state of the system, where $\sum_{i=1}^{N} k_i = 0$, which leads to

$$k_i = -k_{N+1-i} \quad \forall\, i = 1, \ldots, N. \tag{2.6}$$

The k_i are ordered such that $k_N > k_{N-1} > \ldots > k_2 > k_1$. The N (real) numbers k_i are determined as the solution of the Bethe equations. These can be given in many forms [2, 8, 23], we display that of [3]

$$(k_{j+1} - k_j)L = 2\pi + \sum_{i=1}^{N} \left[2\arctan\left(\frac{k_i - k_{j+1}}{c}\right) - 2\arctan\left(\frac{k_i - k_j}{c}\right)\right] \quad j = 1, \ldots, N-1. \tag{2.7}$$

While these equations cannot be solved explicitly for k_j as a function of c, they can be solved for both small and large cL expansions, using the method of quadrature. We list explicit small cL expansions in Appendix A, and at this point we highlight that

$$k_j = \sqrt{\frac{2c}{L}} h_j^{(N)} \left(1 - \frac{1}{24}(cL) + \mathrm{O}(cL)^2\right) \tag{2.8}$$

where $h_j^{(N)}$ is the jth zero of the Nth Hermite polynomial. The leading term in (2.8) is found in reference to this problem by Gaudin [23], for more detail refer to Szegö [24]. The coefficient $-1/24$ in the next term of the expansion is new, and appears to be universal (see Appendix A).

Here we have chosen to display the large cL expansion for k_j in its general N form as given in [8], for its particular utility in the following sections.

$$\begin{aligned} k_j \;=\; & (2j - N - 1)\frac{\pi}{L}\left[1 - 2N\left(\frac{1}{cL}\right) + 4N^2\left(\frac{1}{cL}\right)^2 \right. \\ & \left. + \left\{-8N^3 + \frac{4}{3}N\left[2j^2 + (N+1)(N-2j)\right]\pi^2\right\}\left(\frac{1}{cL}\right)^3\right] + \mathrm{O}\left(\frac{1}{cL}\right)^4. \end{aligned} \tag{2.9}$$

We also list some exact solutions to the Bethe equations in Table I.

N	cL	$k_N L$
2	π	$\pi/2$
3	$\pi/2$	$\pi(\sqrt{17}-3)/4$
3	π	$\sqrt{2}\pi$
3	$3\pi/2$	$3\pi(\sqrt{17}+3)/4$

TABLE I: A sample of exact solutions to (2.7)

It is now possible to explicitly construct the wavefunction $\psi_N(x_1, \ldots, x_N)$ using (2.2) and (2.3). We close here by exhibiting the unnormalised wavefunction $\psi_2(x_1, x_2)$ by way of example

$$\psi_2(x_1, x_2) = \left(1 - \frac{2\mathrm{i}k_2}{c}\right) e^{\mathrm{i}k_2(x_2 - x_1)} - \left(1 + \frac{2\mathrm{i}k_2}{c}\right) e^{-\mathrm{i}k_2(x_2 - x_1)}. \tag{2.10}$$

which has normalisation

$$\mathcal{N}^2 = 2L - \frac{\sin 2k_2 L}{k_2} - \frac{8\sin^2 k_2 L}{c} + \frac{4k_2(2k_2 L + \sin 2k_2 L)}{c^2}. \tag{2.11}$$

In all that follows the weak coupling expansion corresponds to the dimensionless parameter $cL \ll 1$ and the strong coupling expansion corresponds to $cL \gg 1$.

III. DENSITY MATRIX AND OCCUPATION NUMBERS

The normalised density matrix $\rho_N(x, 0)$ is defined as

$$\rho_N(x, 0) = \frac{N}{L}\frac{1}{\mathcal{N}^2}\sum_{j=0}^{N-1}\int_{R_{N-1,j}(x)} dx_1 \ldots dx_{N-1}\psi_N(0, x_1, x_2, \ldots, x_{N-1})\overline{\psi_N(x_1, \ldots, x_j, x, x_{j+1}, \ldots, x_{N-1})} \tag{3.1}$$

where the overbar implies complex conjugation, the normalisation $\mathcal{N}^2$ is specified by (2.4), and the domain of integration is specified by

$$R_{N,j}(x) : 0 \leq x_1 < \ldots < x_j < x < x_{j+1} < \ldots < x_N \leq L. \tag{3.2}$$

The density matrix is normalised such that $\rho_N(0, 0) = \rho_0 = N/L$. Hence to compute the density matrix for N particles, one must perform N lots of $N-1$ dimensional integrals over $(N!)^2$ terms (e.g. 4 triple integrals over 576 terms for $N = 4$). This is a computationally expensive task, and hence we were able to determine solutions only up to $N = 4$ using this method.

The occupation numbers $c_n(N)$ are determined as a Fourier transform of the density matrix

$$c_n(N) = \int_0^L \rho_N(x, 0)e^{2\mathrm{i}\pi nx/L}dx = \int_0^L \rho_N(x, 0)\cos(2\pi nx/L)dx. \tag{3.3}$$

They have the physical interpretation of being the expectation value of the number of particles in mode n.

We note the normalisation property $\sum_{n=-\infty}^{\infty} c_n(N) = N$; we have confirmed this result for all occupation number formulas that follow. It is sometimes useful to discuss occupation number per particle; we introduce the notation $c_n^*(N) = c_n(N)/N$. We shall now display some explicit solutions for $N = 2, 3, 4$ and general N.

A. $N=2$

Within Section III, this subsection heralds the most complete set of results, with results for $N=3$ and 4 becoming increasingly exiguous as the intricacies of the equations develop. For example, it is only possible to display the complete density matrix for $N=2$, as for $N=3$ already the equation would take many pages to display. We have utilised for $N=2$ (2.7) to obtain

$$c = 2k_2 \tan\left(\frac{k_2 L}{2}\right) \tag{3.4}$$

hence producing a concise form of the density matrix using (2.2), (2.3) and (3.1)

$$\rho_2(x,0) = \frac{2}{L}\frac{k_2 x\cos(k_2\,(L-x)) + k_2\,(L-x)\cos(k_2 x) + \sin(k_2\,(L-x)) + \sin(k_2 x)}{k_2 L + \sin(k_2 L)} \qquad 0 \le k_2 \le \pi \tag{3.5}$$

with corresponding occupation numbers from (3.3)

$$c_n(2) = 2\frac{4(k_2 L)^3\,(1-\cos(k_2 L))}{(4n^2\pi^2 - k_2^2 L^2)^2\,(k_2 L + \sin(k_2 L))} \qquad 0 \le k_2 \le \pi. \tag{3.6}$$

The expansion of the density matrix for small cL is given using (A1)

$$\begin{aligned}\rho_2(t,0) &= \frac{2}{L}\Big[1 - \frac{t^2(\pi-t)^2}{24\pi^4}(cL)^2 + \frac{t^2(\pi-t)^2(t^2-\pi t+2\pi^2)}{360\pi^6}(cL)^3 \\ &\quad + \frac{t^2(\pi-t)^2\left(16\pi^4 - 24\pi^3 t + 27\pi^2 t^2 - 6\pi t^3 + 3t^4\right)}{40320\pi^8}(cL)^4 + \mathrm{O}(cL)^5\Big].\end{aligned} \tag{3.7}$$

Note that we have introduced here $t = \pi x/L$, henceforth we switch between t and x as appropriate. The corresponding occupation numbers for (3.7) are given by

$$c_n(2) = 2\begin{cases} 1 - \frac{1}{720}(cL)^2 + \frac{1}{6048}(cL)^3 - \frac{11}{1209600}(cL)^4 + \mathrm{O}(cL)^5 & \text{when } n=0 \\ \frac{1}{16n^4\pi^4}(cL)^2 + \frac{3-n^2\pi^2}{96n^6\pi^6}(cL)^3 + \frac{4n^4\pi^4-30n^2\pi^2+45}{3840n^8\pi^8}(cL)^4 + \mathrm{O}(cL)^5 & \text{when } n\neq 0. \end{cases} \tag{3.8}$$

Utilising (2.9), we obtain a large cL expansion for the density matrix

$$\begin{aligned}\rho_2(t,0) &= \frac{2}{L}\Big\{\frac{(\pi-2t)\cos t + 2\sin t}{\pi} + \frac{8(\pi-t)t\sin t}{\pi}\left(\frac{1}{cL}\right) \\ &\quad + \frac{8\left[(\pi^2-3\pi t+2t^2)t\cos t - (\pi^2+6\pi t-6t^2)\sin t\right]}{\pi}\left(\frac{1}{cL}\right)^2 + \mathrm{O}\left(\frac{1}{cL}\right)^3\Big\}.\end{aligned} \tag{3.9}$$

Note that in the limit $cL \to \infty$, we recover (20) from [10]. The corresponding occupation numbers for (3.9) are given by

$$\begin{aligned}c_n(2) &= 2\Bigg[\frac{8}{(4n^2-1)^2\pi^2} - \frac{32(12n^2+1)}{(4n^2-1)^3\pi^2}\left(\frac{1}{cL}\right) \\ &\quad + \left\{\frac{3072n^2\left(4n^2+1\right)}{\left(4n^2-1\right)^4\pi^2} - \frac{64\left(6n^2-1\right)}{\left(4n^2-1\right)^2}\right\}\left(\frac{1}{cL}\right)^2 + \mathrm{O}\left(\frac{1}{cL}\right)^3\Bigg]\end{aligned} \tag{3.10}$$

which in the limit $cL \to \infty$ recovers (42) from [10]. We also display here the density matrix for the exact solution to (2.7) from Table I, when $cL = \pi, k_2 L = \pi/2$

$$\rho_2(t,0) = \frac{2}{L}\frac{(-t+\pi+2)\cos\frac{t}{2} + (t+2)\sin\frac{t}{2}}{\pi+2} \tag{3.11}$$

and the corresponding occupation numbers are given by

$$c_n(2) = 2\frac{16}{\left(16n^2-1\right)^2\pi(\pi+2)}. \tag{3.12}$$

B. $N = 3$

The small cL expansion of the density matrix is given here, using (3.1) and (A2).

$$\begin{aligned}\rho_3(t,0) &= \frac{3}{L}\Big[1 - \frac{t^2(\pi-t)^2}{12\pi^4}(cL)^2 - \frac{t^2(\pi-t)^2(t^2-\pi t-3\pi^2)}{180\pi^6}(cL)^3 \\ &+ \frac{t^2(\pi-t)^2\left(-101t^4+202\pi t^3-125\pi^2t^2+24\pi^3t+54\pi^4\right)}{20160\pi^8}(cL)^4 + \mathrm{O}(cL)^5\Big].\end{aligned} \tag{3.13}$$

Note that the coefficient of the $(cL)^p$ term is a polynomial in t of order $2p$, for $p \geq 2$. This structure is also repeated in (3.7).

The corresponding occupation numbers for (3.13) are given by

$$c_n(3) = 3\begin{cases} 1 - \frac{1}{360}(cL)^2 + \frac{1}{1680}(cL)^3 - \frac{163}{1814400}(cL)^4 + \mathrm{O}(cL)^5 & \text{when } n = 0 \\ \frac{1}{8n^4\pi^4}(cL)^2 - \frac{n^2\pi^2+3}{48n^6\pi^6}(cL)^3 + \frac{6n^4\pi^4+170n^2\pi^2-1515}{1920n^8\pi^8}(cL)^4 + \mathrm{O}(cL)^5 & \text{when } n \neq 0. \end{cases} \tag{3.14}$$

We give here the large cL expansion of the density matrix, using (3.1) and (2.7)

$$\begin{aligned}\rho_3(t,0) &= \frac{3}{L}\Big\{\frac{2(\pi-2t)^2+12(\pi-2t)\sin 2t+4(2t-\pi+2)(2t-\pi-2)\cos 2t+\cos 4t+15}{6\pi^2} \\ &+ \frac{4\sin t\left[(\pi-2t)\left(1+8\pi t-8t^2\right)\cos t-(\pi-2t)\cos 3t+8\left(\pi-t\right)t\sin t\right]}{\pi^2}\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2\Big\}\end{aligned} \tag{3.15}$$

which in the limit $cL \to \infty$ recovers (21) of [10].

The occupation numbers corresponding to (3.15) are given by

$$c_n(3) = 3\begin{cases} \left(\frac{1}{9}+\frac{35}{6\pi^2}\right) + \left(\frac{8}{3}+\frac{35}{\pi^2}\right)\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } n = 0 \\ \frac{1}{9} - \left(\frac{4}{3}+\frac{35}{6\pi^2}\right)\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n| = 1 \\ \frac{35}{108\pi^2} - \frac{385}{36\pi^2}\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n| = 2 \\ \frac{2(3n^2+1)}{3n^2(n^2-1)^2\pi^2} - \frac{4\left(9n^6-28n^4-61n^2+8\right)}{n^2(n^2-1)^3(n^2-4)\pi^2}\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n| \geq 3. \end{cases} \tag{3.16}$$

which in the limit $cL \to \infty$ recovers (45) of [10]. We also display here the density matrix for the exact solution to (2.7) from Table I, when $cL = \sqrt{2}\pi, k_3L = \pi$

$$\rho_3(t,0) = \frac{3}{L}\Big\{\frac{8t^2-8\pi t+2\phi(\pi-2t)\cos t-4\cos 2t+4\left[\pi\left(2\sqrt{2}t+5\right)-2\sqrt{2}\left(t^2-5\right)\right]\sin t+\phi\pi+52}{48+3\phi\pi}\Big\} \tag{3.17}$$

where $\phi = 10\sqrt{2}+3\pi$. The corresponding occupation numbers for (3.17) are given by

$$c_n(3) = 3\begin{cases} \frac{5}{27} + \frac{4\sqrt{2}}{\pi} - \frac{4(299+71\sqrt{2}\pi)}{27(16+\phi\pi)} & \text{when } n = 0 \\ \frac{64\sqrt{2}+54\pi}{81\pi(16+\phi\pi)} & \text{when } |n| = 1 \\ \frac{4\left(-\pi+32n^4\left(2\sqrt{2}+\pi\right)-4n^2\left(12\sqrt{2}+\pi\right)\right)}{3n^2\pi(4n^2-1)^3(16+\phi\pi)} & \text{when } |n| \geq 2 \end{cases} \tag{3.18}$$

The density matrix and occupation numbers for the other exact values listed in Table I for $N = 3$, although calculated, are too lengthy to display here.

C. $N = 4$

The intricacy of the density matrix at this value of N is already so great that we go on to calculate the occupation numbers without explicitly exhibiting it, and as such we give only $c_0(4)$. We utilise (3.3) and (A3) to produce the result

$$c_0(4) = 4\Big[1 - \frac{1}{240}(cL)^2 + \frac{13}{10080}(cL)^3 - \frac{383}{1209600}(cL)^4 + \mathrm{O}(cL)^5\Big]. \tag{3.19}$$

D. General N

It is interesting to observe that with the presence of the irrational numbers in the Bethe equations for small cL (Appendix A), that when the final results for the occupation numbers appear they contain purely rational numbers. Encouraged by this remarkable observation and other indications in the preceding subsections, we looked for a pattern in N for the coefficients in the small cL expansion for a general $c_0(N)$. Upon close examination of (3.6), (3.14) and (3.19), and with some good fortune, we found the following polynomial structure for the small cL expansion of the $n = 0$ occupation number for general N.

$$\begin{aligned} c_0^*(N) \;&=\; 1 - \frac{N-1}{720}(cL)^2 + \frac{N-1}{720}\left(\frac{4(N-1)+1}{42}\right)(cL)^3 \\ &\quad - \frac{N-1}{720}\frac{1}{42}\left(\frac{45(N-1)^2 - 5(N-1) - 7}{120}\right)(cL)^4 + \mathrm{O}(cL)^5. \end{aligned} \tag{3.20}$$

We have written (3.20) as given to emphasise the detailed structure of the coefficients in this expansion. Note that the coefficient of $(N-1)(cL)^p$ is a polynomial of order $p-2$ in $N-1$ for $p \geq 2$. Therefore to obtain say the coefficient of the $(cL)^5$ term we would need $c_0(N)$ for four specific values of N in their rational form. Unfortunately, it was too computationally expensive for us to obtain any $N-1$ polynomial for any higher order than $(cL)^4$. We conjecture that this is the pattern for all $N \geq 2$.

The fact that the coefficient of $(N-1)(cL)^p$ appears to be a polynomial of degree $p-2$ in $N-1$ suggests that for any finite N there is always an interval $cL \in [0, D_N)$ such that the series is convergent, but with $D_N \to 0$ as $N \to \infty$. To quantify this last point, note that in the thermodynamic limit the dimensionless parameter is $c/\rho_0 = \Gamma$ and the coefficient is proportional to N^{2p-1}, which suggests that the corresponding radius of convergence is proportional to $1/N^2$. In particular this means that no information can be gleaned as to the functional form of $c_0^*(N)$ as a function of Γ about Γ=0 except that it is not analytic.

IV. DENSITY MATRICES AND OCCUPATION NUMBERS FOR LARGE cL

In the previous section, we gave large cL expansions for the density matrices and occupation numbers for $N = 2$ and $N = 3$. Again, it was numerically prohibitive to go beyond $N = 4$, furthermore as can be seen from the coefficients in these expansions (and as can be witnessed in the Tables III-VIII which we will refer to in what follows), that the numbers are highly irrational with no hope of finding an analogous pattern as we were fortunate enough to do in (3.20).

We therefore turn to a totally different mathematical strategy to obtain a large cL expansion for these quantities. In the impenetrable limit ($cL = \infty$, arbitrary L), Lenard [7] developed the theory for the density matrix for arbitrary N, and went on to show that for asymptotically large N that $c_0(N) \sim N^{1/2}$.

In our recent work [10] we employed Lenard's theory to obtain the results for the occupation numbers for a range of finite N, and from them determine the results for general N, which continue on to the asymptotically large N limit [31].

To go beyond the impenetrable limit for these quantities, we turn to the very valuable work of Jimbo and Miwa [8]. Building upon the work of Lenard [7], they developed an expansion for the density matrix in the large cL limit for general N in principle. We say in principle because while their expansion is superb in the form given, it is as numerically prohibitive to use as was the method we employed in the previous section.

To resolve this difficulty we have recast their theory using mathematical techniques evidenced in our recent work [12] and presented in great detail by Forrester [25] into a new form that is readily amenable to numerical calculation.

We will find in what follows that the specific results for $N = 2$ and $N = 3$ as in Section III are useful specific checks to the theory.

In Subsection IV A along with Appendix B, we present the full details of our derivation for the the density matrix. Following upon that, in Subsection IV B we are now able to calculate the occupation numbers for a finite range of N values and from these results are able to determine the results for general N which again continue to asymptotically large N.

A. Toeplitz Determinants

The fact that $|\psi_N|^2$ consists of $(N!)^2$ terms means any method based on term-by-term integration must necessarily be restricted to small N. To overcome this one must seek out structure in the form of ψ_N, and this structure must be used to reduce the computational expense required to compute $\rho_N(x,0)$. Certainly in the limit $cL \to \infty$ there is structure in (2.2) for then (2.9) gives $k_j = (2j - N - 1)\pi/L$ while (2.3) gives $a(p) = 1$, and so

$$\begin{aligned} \psi_N(x_1,\dots,x_N)|_{cL\to\infty} &= \sum_{p\in S_N} \epsilon(p) \prod_{j=1}^{N} e^{\mathrm{i}\pi(2j-N-1)x_{p(j)}/L} && (4.1)\\ &= \det[e^{\mathrm{i}\pi(2k-N-1)x_j/L}]_{j,k=1,\dots,N}. && (4.2) \end{aligned}$$

While the sum consists of $N!$ terms, the determinant can be computed in $\mathrm{O}(N^3)$ arithmetic operations. Moreover the corresponding density matrix can also be expressed as a determinant (Lenard [7])

$$\rho_N^{(0)}(x,0) = -\frac{1}{2}\Delta_1\!\left(\begin{matrix} x \\ 0 \end{matrix}; -2\right) \tag{4.3}$$

where

$$\Delta_1\!\left(\begin{matrix} x \\ 0 \end{matrix}; \lambda\right) = \lambda e^{-\mathrm{i}\pi(N-1)x/L} \det\left[A_1(j-k)\right]_{j,k=1,\dots,N-1} \tag{4.4}$$

$$\begin{aligned} A_1(j-k) &= \frac{1}{L}\left(\int_0^L + \lambda \int_0^x\right) du (e^{2\pi\mathrm{i}u/L} - e^{2\pi\mathrm{i}x/L})(e^{-2\pi\mathrm{i}u/L} - 1)e^{2\pi\mathrm{i}u(j-k)/L} && (4.5)\\ &= \begin{cases} \frac{\lambda}{2\pi}\sin 2t - \frac{t\lambda}{\pi} - 1 & \text{when } j-k=-1 \\ 2e^{\mathrm{i}t}\left[(\frac{t\lambda}{\pi}+1)\cos t - \frac{\lambda}{\pi}\sin t\right] & \text{when } j-k=0 \\ e^{2\mathrm{i}t}\left[\frac{\lambda}{2\pi}\sin 2t - \frac{t\lambda}{\pi} - 1\right] & \text{when } j-k=1 \\ \frac{2\lambda\mathrm{i}e^{\mathrm{i}(j-k+1)t}}{(j-k)((j-k)^2-1)\pi}\left[(j-k)\cos((j-k)t)\sin t - \cos t\sin((j-k)t)\right] & \text{when } |j-k|\geq 2 \end{cases} && (4.6) \end{aligned}$$

and thus also can be computed in $\mathrm{O}(N^3)$ operations. In [10] this determinant formula was used to compute the density matrix and the corresponding ground state occupation numbers up to $N = 7$. It has been shown by Jimbo and Miwa [8] (see also Section VI below) that determinant structures persist if ψ_N is expanded in large cL, and that this implies special structures for the expansion of the density matrix. In particular, writing

$$\rho_N(x,0) = \rho_N^{(0)}(x,0) + \left(\frac{1}{cL}\right)\rho_N^{(1)}(x,0) + \mathrm{O}\left(\frac{1}{cL}\right)^2 \tag{4.7}$$

it was shown in [8] that

$$\rho_N^{(1)}(x,0) = -2\rho_0 x \frac{\partial}{\partial x}\rho_N^{(0)}(x,0) + F_N(x) \tag{4.8}$$

where $\rho_N^{(0)}(x,0)$ is specified by (4.3) and

$$F_N(x) = \frac{1}{\Delta(x;-2)}\left[\frac{\partial}{\partial x}\Delta_1\begin{pmatrix} x \\ 0 \end{pmatrix}; \lambda\Bigg) \frac{\partial}{\partial\lambda}\Delta(x;\lambda) + \frac{\partial}{\partial\lambda}\Delta_1\begin{pmatrix} x \\ 0 \end{pmatrix}; \lambda\Bigg) \frac{\partial}{\partial x}\Delta(x;\lambda) - \Delta_1\begin{pmatrix} x \\ 0 \end{pmatrix}; \lambda\Bigg) \frac{\partial^2}{\partial x\partial\lambda}\Delta(x;\lambda)\right]\Bigg|_{\lambda=-2} . \tag{4.9}$$

Introducing the kernel function

$$K_{NL}(x,y) = \frac{1}{L}\sum_{j=1}^{N} e^{-\mathrm{i}\pi(2j-N-1)(x-y)/L} \tag{4.10}$$

$$= \frac{\sin[N\pi(x-y)/L]}{L\sin[\pi(x-y)/L]} \tag{4.11}$$

the quantity $\Delta_1\left(\begin{matrix} x \\ y \end{matrix}; \lambda\right)$ is defined in [8] as the Fredholm minor

$$\Delta_1\left(\begin{matrix} x \\ y \end{matrix}; \lambda\right) = \sum_{l=0}^{\infty}\frac{\lambda^{l+1}}{l!}\int_0^x du_1 \ldots \int_0^x du_l \times \det\begin{bmatrix} K_{NL}(x,y) & [K_{NL}(x,u_k)]_{k=1,\ldots,l} \\ [K_{NL}(u_j,y)]_{j=1,\ldots,l} & [K_{NL}(u_j,u_k)]_{j,k=1,\ldots,l} \end{bmatrix} \tag{4.12}$$

while $\Delta(x;\lambda)$ is specified as the Fredholm determinant

$$\Delta(x;\lambda) = \sum_{l=0}^{\infty}\frac{\lambda^l}{l!}\int_0^x du_1 \ldots \int_0^x du_l \det\left[K_{NL}(u_j,u_k)\right]_{j,k=1,\ldots,l} . \tag{4.13}$$

Neither (4.12) or (4.13) are suitable for computation. However for Δ_1 we have the determinant formula (4.4), and Δ too can be expressed as a determinant. This can be seen by developing Lenard's [7] derivation of (4.3), which proceeds by writing the summation of determinants of (4.12) and (4.13) as multidimensional integrals of a type which can be recognised as Toeplitz determinants (see Appendix B). In addition to (4.4) - (4.6), one deduces that

$$\Delta(x;\lambda) = \det\left[A_0(j-k)\right]_{j,k=1,\ldots,N} \tag{4.14}$$

where

$$A_0(j-k) = \frac{1}{L}\left(\int_0^L + \lambda\int_0^x\right) du \exp[2\pi\mathrm{i}u(j-k)/L] \tag{4.15}$$

$$= \begin{cases} 1+\frac{\lambda x}{L} & \text{when } j-k=0 \\ \frac{\lambda}{2\mathrm{i}\pi(j-k)}(e^{2\mathrm{i}\pi(j-k)x/L}-1) & \text{when } j-k\neq 0. \end{cases} \tag{4.16}$$

With these determinant formulas $F_N(x)$ is expressed in a computable form. This concludes the method necessary to construct the density matrix expanded in large cL. We now examine the occupation numbers.

B. Occupation Numbers for Large cL

The following notation for the occupation number $c_n(N)$ expanded in large cL is given as

$$c_n(N) = c_n^{(0)}(N) + \left(\frac{1}{cL}\right)c_n^{(1)}(N) + \mathrm{O}\left(\frac{1}{cL}\right)^2 . \tag{4.17}$$

We list some exact values of $c_n^{(0)}(N)$ and $c_n^{(1)}(N)$ for $n = 0$ ($N = 2, \ldots, 7$) and $n = 1, 2$ ($N = 2, \ldots, 6$) in Tables III, IV and V. The $c_n^{(1)}(N)$ term can be expanded further as

$$\begin{aligned} c_n^{(1)}(N) &= \int_0^L \left(-2Nx \frac{\partial \rho_N^{(0)}(x,0)}{\partial x} + F_N(x) \right) \cos\left(\frac{2\pi n x}{L}\right) dx & (4.18)\\ &= c_n^{(1,1)}(N) + c_n^{(1,2)}(N) & (4.19) \end{aligned}$$

where $F_N(x)$ may be computed using the expressions in Subsection IV A above. The $c_n^{(1,1)}(N)$ term can be simplified using integration by parts to

$$c_n^{(1,1)}(N) = -2N^2 + 2Nc_n^{(0)}(N) - 4\pi n N \int_0^L \frac{x}{L} \sin(2\pi n x/L) \rho_n^{(0)}(x) dx. \tag{4.20}$$

We list some exact values of $c_n^{(1,1)}(N)$ and $c_n^{(1,2)}(N)$ for $n = 0, 1, 2$ with $N = 2, \ldots, 6$ in Tables VI, VII and VIII. We present all the numerical values that we computed for $c_n^{*(0)}(N)$, $c_n^{*(1,1)}(N)$, $c_n^{*(1,2)}(N)$, and $c_n^{*(1)}(N)$ obtained for $n = 0, 1, 2$ in Appendix C. In this table, the data is given to 6 significant figures for economy of presentation, while in point of fact, accuracy to 10 significant figures was needed for the analysis that now follows. We were able to achieve this numerical accuracy up to $N = 36$ for the $n = 0$ mode, and up to $N = 26$ for the modes $n = 1$ and $n = 2$. Note that $c_n^{*(0)}(N)$ for $n = 0, 1, 2$ reproduces the results of [10].

We now introduce a generalised version of the ansatz first introduced in [10], expanded to $\mathrm{O}(1/cL)$

$$\begin{aligned} c_n(N) &= A_{\infty,n}\left(1 + \frac{\alpha_n N}{cL}\right) N^{\frac{1}{2} + \frac{\beta_n N}{cL}} + C_{\infty,n}\left(1 + \frac{\gamma_n N}{cL}\right) & (4.21)\\ &= (A_{\infty,n}\sqrt{N} + C_{\infty,n}) + N(A_{\infty,n}\alpha_n\sqrt{N} + A_{\infty,n}\beta_n\sqrt{N}\ln N + C_{\infty,n}\gamma_n)\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2. & (4.22) \end{aligned}$$

It is now pertinent to determine bounds on α_n and β_n, and a value for γ_n. Consider the set of three linear equations describing the $1/cL$ term of (4.22) for three consecutive values of N

$$\begin{pmatrix} \sqrt{N-1} & \sqrt{N-1}\ln(N-1) & 1 \\ \sqrt{N} & \sqrt{N}\ln(N) & 1 \\ \sqrt{N+1} & \sqrt{N+1}\ln(N+1) & 1 \end{pmatrix} \begin{pmatrix} A_{\infty,n}\alpha_n \\ A_{\infty,n}\beta_n \\ C_{\infty,n}\gamma_n \end{pmatrix} = \begin{pmatrix} c_n^{*(1)}(N-1) \\ c_n^{*(1)}(N) \\ c_n^{*(1)}(N+1) \end{pmatrix} \tag{4.23}$$

and the set of two linear equations describing the $1/cL$ term of (4.22) with $C_{\infty,n} = 0$ for two consecutive values of N

$$\begin{pmatrix} \sqrt{N} & \sqrt{N}\ln(N) \\ \sqrt{N+1} & \sqrt{N+1}\ln(N+1) \end{pmatrix} \begin{pmatrix} A_{\infty,n}\alpha_n \\ A_{\infty,n}\beta_n \end{pmatrix} = \begin{pmatrix} c_n^{*(1)}(N) \\ c_n^{*(1)}(N+1) \end{pmatrix}. \tag{4.24}$$

Values of $A_{\infty,n}$ and $C_{\infty,n}$ from [10] are displayed in Table II. The solution for α_n, β_n, γ_n, of (4.23) and the solution for α_n, β_n for (4.24) for various values of N establish bounds on α_n and β_n, and a value for γ_n. We establish numerical stability for these bounds by calculating them for $N = 2$ up to $N = 35$. A fractional accuracy of $\gtrsim 10^{-10}$ is necessary, and hence the parameters are calculated at $N = 35$ for $n = 0$, and $N = 25$ for $n = 1$ and $n = 2$. We present them in Table II.

Note that β_n is very close to 2 for $n = 0$, and suggestive of the value 2 for $n = 1$ and 2. We postulate that $\beta_n = 2$ for all n. We say more about this β_n in Section VII.

n	$A_{\infty,n}$	$C_{\infty,n}$	α_n	β_n	γ_n
0	1.54273	-0.5725	$0.1561 < \alpha_0 < 0\ .1838$	$1.998 < \beta_0 < 2.003$	0.1599
1	0.5143	-0.5739	$-5.709 < \alpha_1 < -5.067$	$1.972 < \beta_1 < 2.094$	-1.109
2	0.3676	-0.5775	$-8.350 < \alpha_2 < -6.121$	$1.887 < \beta_2 < 2.313$	-2.736

TABLE II: Parameters for the ansatz (4.22)

V. CORRELATION FUNCTIONS AND STRUCTURE FACTORS

The definition of the two-point correlation function $g_N(x,0)$ is

$$g_N(x,0) = \frac{1}{\mathcal{N}^2} \sum_{j=0}^{N-2} \int_{R_{N-2,j}(x)} dx_1 \dots dx_{N-2} |\psi_N(0, x_1, \dots, x_j, x, x_{j+1}, \dots, x_{N-2})|^2 \tag{5.1}$$

where the normalisation $\mathcal{N}^2$ is specified by (2.4), and the domain of integration by (3.2). Then (5.1) has the property

$$\int_0^L g_N(x,0)dx = N-1, \tag{5.2}$$

in keeping with the interpretation of $g_N(x,0)$ as the density of particles at position x, given there is a particle at the origin. The structure factor is defined by

$$\mathcal{S}_n(N) = \frac{1}{N}\frac{1}{\mathcal{N}^2} \int_{R_{N-1,j}} dx_1 \dots dx_{N-1} \left| \sum_{j=0}^{N-1} e^{2\mathrm{i}\pi x_j n/L} \right|^2 |\psi_N(0, x_1, \dots, x_{N-1})|^2 \tag{5.3}$$

where $x_0 = 0$, which can be written in terms of $g_N(x,0)$ to read

$$\mathcal{S}_n(N) = 1 + \int_0^L g_N(x,0)\cos(2\pi n x/L)dx. \tag{5.4}$$

In view of (5.2) this gives $\mathcal{S}_0(N) = N$.

A. $N = 2$

This subsection yields the most complete set of results within Section V, with the set of results for $N = 3$ and $N = 4$ becoming increasingly limited. For $N = 4$, we display only the small cL expansion of the correlation function and structure factor.

It is possible to produce a simple, exact form of the correlation function for $N = 2$ by utilising (3.4) to obtain

$$g_2(x,0) = \frac{2k_2\cos^2[\frac{1}{2}k_2(L-2x)]}{k_2L + \sin(k_2L)} \tag{5.5}$$

which has corresponding structure factor

$$\mathcal{S}_n(2) = \begin{cases} 2 & \text{when } n = 0 \\ 1 + \frac{k_2^2L^2\sin(k_2L)}{\left(k_2^2L^2 - n^2\pi^2\right)(k_2L + \sin(k_2L))} & \text{when } n \neq 0. \end{cases} \tag{5.6}$$

The correlation function, expanded about small cL using (A1), is given by

$$\begin{aligned} g_2(x,0) &= \frac{1}{L}\Big[1+\left(-\frac{x^2}{L^2}+\frac{x}{L}-\frac{1}{6}\right)(cL)+\left(\frac{x^4}{3L^4}-\frac{2x^3}{3L^3}+\frac{x^2}{2L^2}-\frac{x}{6L}+\frac{1}{60}\right)(cL)^2 \\ &\quad +\left(-\frac{2x^6}{45L^6}+\frac{2x^5}{15L^5}-\frac{7x^4}{36L^4}+\frac{x^3}{6L^3}-\frac{7x^2}{90L^2}+\frac{x}{60L}-\frac{1}{945}\right)(cL)^3+\mathrm{O}(cL)^4\Big] \end{aligned} \tag{5.7}$$

and hence the structure factor is given by

$$\mathcal{S}_n(2)=\begin{cases} 2 & \text{when } n=0 \\ 1-\frac{1}{2n^2\pi^2}(cL)+\frac{n^2\pi^2-6}{12n^4\pi^4}(cL)^2-\frac{n^4\pi^4-15n^2\pi^2+60}{120n^6\pi^6}(cL)^3+\mathrm{O}(cL)^4 & \text{when } n\neq 0. \end{cases} \tag{5.8}$$

The large cL expansion of the correlation function is given using (2.9)

$$g_2(t,0)=\frac{1}{L}\left\{2\sin^2 t+8\sin t[(\pi-2t)\cos t-\sin t]\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2\right\} \tag{5.9}$$

taking the Fourier transform then yields the structure factor

$$\mathcal{S}_n(2)=\begin{cases} 2 & \text{when } n=0 \\ \frac{1}{2}+3\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n|=1 \\ 1-\frac{4}{(n^2-1)}\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n|\geq 2. \end{cases} \tag{5.10}$$

The correlation function for the special values $cL=\pi$, $k_2L=\pi/2$, (see Table I) is

$$g_2(t,0)=\frac{\pi\,(\sin t+1)}{L(\pi+2)} \tag{5.11}$$

and hence the structure factor is

$$\mathcal{S}_n(2)=\begin{cases} 2 & \text{when } n=0 \\ 1-\frac{2}{(4n^2-1)(\pi+2)} & \text{when } n\neq 0. \end{cases} \tag{5.12}$$

B. $N=3$

Given the length of the full correlation function for $N=3$, it is not possible to display the full result here, however we shall display the small and large cL expansions for both the correlation function and the structure factor.

The small cL expansion of the correlation function is given, using (A2)

$$\begin{aligned} g_3(x,0) &= \frac{2}{L}\Big[1+\left(-\frac{x^2}{L^2}+\frac{x}{L}-\frac{1}{6}\right)(cL)+\left(\frac{x^4}{12L^4}-\frac{x^3}{6L^3}+\frac{x^2}{4L^2}-\frac{x}{6L}+\frac{1}{40}\right)(cL)^2 \\ &\quad +\left(\frac{x^6}{5L^6}-\frac{3x^5}{5L^5}+\frac{5x^4}{8L^4}-\frac{x^3}{4L^3}+\frac{x}{40L}-\frac{1}{280}\right)(cL)^3+\mathrm{O}(cL)^4\Big] \end{aligned} \tag{5.13}$$

and hence structure factor by

$$\mathcal{S}_n(3)=\begin{cases} 3 & \text{when } n=0 \\ 1-\frac{1}{n^2\pi^2}(cL)+\frac{2n^2\pi^2-3}{12n^4\pi^4}(cL)^2-\frac{n^4\pi^4+15n^2\pi^2-180}{40n^6\pi^6}(cL)^3+\mathrm{O}(cL)^4 & \text{when } n\neq 0. \end{cases} \tag{5.14}$$

The large cL expansion for the correlation function is given using (2.9)

$$g_3(t,0) = \frac{2}{L}\left\{\frac{4\,(2+\cos 2t)\sin^2 t}{3} - \frac{8\sin t\,[-2\,(\pi-2t)\,(2\cos t+\cos 3t)+3\sin t+4\sin 3t]}{3}\left(\frac{1}{cL}\right) + \mathrm{O}\left(\frac{1}{cL}\right)^2\right\} \tag{5.15}$$

which yields the structure factor

$$\mathcal{S}_n(3) = \begin{cases} 3 & \text{when } n=0 \\ \frac{1}{3}+\frac{32}{9}\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n|=1 \\ \frac{2}{3}+\frac{38}{9}\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n|=2 \\ 1-\frac{16(n^2-2)}{(n^2-4)(n^2-1)}\left(\frac{1}{cL}\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2 & \text{when } |n|\geq 3. \end{cases} \tag{5.16}$$

We now display the correlation function for the special value $k_3L=\pi$, $cL=\sqrt{2}\pi$, (from Table I)

$$g_3(t,0) = -\frac{2\pi\left[2\left(4\sqrt{2}-\pi\right)\cos t-\pi\cos 2t-4\left(2+\sqrt{2}\pi\right)\sin t+2\left(6+\sqrt{2}\pi\right)\sin 2t-9\pi-8\sqrt{2}\right]}{L\left(16+16\sqrt{2}\pi+9\pi^2\right)} \tag{5.17}$$

and the corresponding structure factor by

$$\mathcal{S}_n(3) = \begin{cases} 3 & \text{when } n=0 \\ 1-\frac{32+16\sqrt{2}\pi-3\pi^2}{48+48\sqrt{2}\pi+27\pi^2} & \text{when } |n|=1 \\ 1-\frac{16\left(2+\sqrt{2}\pi\right)}{(4n^2-1)\left(16+16\sqrt{2}\pi+9\pi^2\right)} & \text{when } |n|\geq 2. \end{cases} \tag{5.18}$$

C. $N=4$

Here we display the correlation function for the small cL expansion of $N=4$

$$\begin{aligned} g_4(x,0) &= \frac{3}{L}\Big[1+\left(-\frac{x^2}{L^2}+\frac{x}{L}-\frac{1}{6}\right)(cL)+\left(-\frac{x^4}{6L^4}+\frac{x^3}{3L^3}-\frac{x}{6L}+\frac{1}{30}\right)(cL)^2 \\ &\quad+\left(\frac{7x^6}{18L^6}-\frac{7x^5}{6L^5}+\frac{47x^4}{36L^4}-\frac{2x^3}{3L^3}+\frac{19x^2}{180L^2}+\frac{x}{30L}-\frac{1}{135}\right)(cL)^3+\mathrm{O}(cL)^4\Big] \end{aligned} \tag{5.19}$$

and its corresponding structure factor

$$\mathcal{S}_n(4) = \begin{cases} 4 & \text{when } n=0 \\ 1-\frac{3}{2n^2\pi^2}(cL)+\frac{n^2\pi^2+3}{4n^4\pi^4}(cL)^2-\frac{2n^4\pi^4+60n^2\pi^2-525}{40n^6\pi^6}(cL)^3+\mathrm{O}(cL)^4 & \text{when } n\neq 0. \end{cases} \tag{5.20}$$

D. General N

Through close examination of the small cL expansions of the correlation function for $N=2$ (5.7), $N=3$ (5.13) and $N=4$ (5.19), we find the following polynomial structure

$$\begin{aligned} g_N(x,0) &= \frac{N-1}{L}\Big\{1-f(x)(cL)+\left[\left(-\frac{N}{4}+\frac{5}{6}\right)f(x)^2+\left(\frac{N}{12}-\frac{1}{9}\right)f(x)+\left(\frac{N}{720}-\frac{1}{216}\right)\right](cL)^2 \\ &\quad+\Big[\left(-\frac{N^2}{36}+\frac{23N}{60}-\frac{7}{10}\right)f(x)^3+\left(\frac{N^2}{36}-\frac{7N}{40}+\frac{1}{5}\right)f(x)^2 \\ &\quad+\left(-\frac{N^2}{144}+\frac{13N}{720}-\frac{1}{120}\right)f(x)+\left(-\frac{N^2}{6804}+\frac{79N}{90720}-\frac{1}{1080}\right)\Big](cL)^3+\mathrm{O}(cL)^4\Big\} \end{aligned} \tag{5.21}$$

where $f(x) = x^2/L^2 - x/L + 1/6$. We conjecture that this structure continues for all N. This has corresponding structure factor

$$\mathcal{S}_n(N) = \begin{cases} N & \text{when } n = 0 \\ 1 - \frac{(N-1)}{2n^2\pi^2}(cL) + \frac{(N-1)\left(2n^2\pi^2+9N-30\right)}{24n^4\pi^4}(cL)^2 & \\ \quad -\frac{(N-1)\left(Nn^4\pi^4+75Nn^2\pi^2-180n^2\pi^2+75N^2-1035N+1890\right)}{240n^6\pi^6}(cL)^3 + \mathrm{O}(cL)^4 & \text{when } n \neq 0 \end{cases} \tag{5.22}$$

We have not proceeded further than $N = 3$ for the large cL expansion for the same lack of utility as we encountered in the case of the density matrix in Section III. There is now, once again, a powerful way to proceed to develop such an expansion for general N, and we present the derivation in the following section. The special case of $N = 2$ (5.9) and $N = 3$ (5.15) provide useful checks.

In closing this section, we note that in (5.21), the leading order term for $g_N(0,0)$ is $(N-1)/L$, which is the free Bose result. Deviations from this begin at order cL.

VI. CORRELATION FUNCTIONS AND STRUCTURE FACTORS FOR LARGE cL

In Section IV A results from Jimbo and Miwa [8] were used to express the $\mathrm{O}(1/cL)$ correction to the density matrix in the form of Toeplitz determinants, which could then be numerically analysed. The method of [8] can also be applied to the calculation of the $\mathrm{O}(1/cL)$ correction to the two-point correlation function (5.1), yielding in fact a closed form analytic expression. To derive this, we follow the working in [8], and begin by noting that the $\mathrm{O}(1/cL)$ expansion of (2.2) is

$$\psi_N(x_1, x_2, \ldots, x_N) = \left(1 + \frac{1}{cL}\left\{\sum_{l=1}^{N}(-N+2l-1)\frac{\partial}{\partial x_l} - 2\rho_0\sum_{l=1}^{N} x_l \frac{\partial}{\partial x_l}\right\} + \ldots\right) \det[e^{\mathrm{i}k_j x_l}] \tag{6.1}$$

Now put $N \to N+2$ (for convenience) and label the particles as

$$0 < y < x_1 < \ldots < x_j < x < x_{j+1} < \ldots < x_N < L. \tag{6.2}$$

In the definition of $g_{N+2}(x,0)$, x will be the variable as in (6.2) and we will take $y \to 0$. With these labels and $N \to N+2$ the operator in (6.1) reads

$$-(N+1)\frac{\partial}{\partial y} + \sum_{l=1}^{j}(-N-1+2l)\frac{\partial}{\partial x_l} + (-N+1+2j)\frac{\partial}{\partial x} + \sum_{l=j+1}^{N}(-N+1+2l)\frac{\partial}{\partial x_l} - 2\rho_0\left(\sum_{l=1}^{N} x_l\frac{\partial}{\partial x_l} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right). \tag{6.3}$$

The determinant in (6.1) has the translation invariance property

$$\det\begin{bmatrix} e^{\mathrm{i}k_j y} \\ e^{\mathrm{i}k_j x_k} \end{bmatrix} = \det\begin{bmatrix} 1 \\ e^{\mathrm{i}k_j (x_k - y)} \end{bmatrix} \tag{6.4}$$

since $\sum k_j = 0$. This means that we can write

$$\frac{\partial}{\partial y} = -\frac{\partial}{\partial x} - \sum_{j=1}^{N}\frac{\partial}{\partial x_j} \tag{6.5}$$

and using too the fact that we want $y \to 0$, the operator (6.3) reads

$$\sum_{l=1}^{j} 2l\frac{\partial}{\partial x_l} + 2(1+j)\frac{\partial}{\partial x} + \sum_{l=j+1}^{N} 2(1+l)\frac{\partial}{\partial x_l} - 2\rho_0\left(\sum_{l=1}^{N} x_l\frac{\partial}{\partial x_l} + x\frac{\partial}{\partial x}\right). \tag{6.6}$$

In keeping with (5.1), by definition

$$g_{N+2}(x,0)=\frac{1}{\mathcal{N}^2}\lim_{y\to 0}\sum_{j=0}^{N}\int_{R_{N,j}(y,x)} dx_1\ldots dx_N\,|\psi_{N+2}(y,x_1,\ldots,x_j,x,x_{j+1},\ldots,x_N)|^2 \tag{6.7}$$

where $\mathcal{N}^2$ is the normalisation, defined by (2.4), and $R_{N,j}(y,x)$ is the region of integration specified by

$$R_{N,j}(y,x): 0\leq y\leq x_1\leq\ldots\leq x_j\leq x\leq x_{j+1}\leq\ldots\leq x_N\leq L. \tag{6.8}$$

With

$$(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{N+2})=(0,x_1,\ldots,x_j,x,x_{j+1},\ldots,x_N) \tag{6.9}$$

and

$$A = \sum_{l=1}^{N} x_l\frac{\partial}{\partial x_l}+x\frac{\partial}{\partial x} \tag{6.10}$$

$$B_j = \sum_{l=1}^{j} 2l\frac{\partial}{\partial x_l}+2(1+j)\frac{\partial}{\partial x}+\sum_{l=j+1}^{N}2(1+l)\frac{\partial}{\partial x_l} \tag{6.11}$$

we have

$$\begin{aligned}\lim_{y\to 0}|\psi_{N+2}(y,x_1,\ldots,x_j,x,x_{j+1},\ldots,x_N)|^2 &= |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2-\frac{2\rho_0}{c}A\left(|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\right)\\ &\quad+\frac{1}{c}B_j\left(|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\right)+\mathrm{O}\left(\frac{1}{cL}\right)^2.\end{aligned} \tag{6.12}$$

The normalisation (2.4) was first expanded in large cL by [8], we display here the first two orders

$$\mathcal{N}^2=(\mathcal{N}^{(\infty)})^2\left[1+\frac{2\rho_0}{c}(N+1)+\mathrm{O}\left(\frac{1}{cL}\right)^2\right]. \tag{6.13}$$

where $(\mathcal{N}^{(\infty)})^2$ is the normalisation (2.4) in the limit $cL\to\infty$. Substituting (6.12) and (6.13) in (6.7) shows that to $\mathrm{O}(1/cL)$

$$g_{N+2}(x,0) = g^{(\infty)}_{N+2}(x,0)-\frac{2\rho_0}{c}(N+1)g^{(\infty)}_{N+2}(x,0) \tag{6.14}$$

$$-\frac{2\rho_0}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\sum_{j=0}^{N}\int_{R_{N,j}(x)}dx_1\ldots dx_N A\left(|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\right) \tag{6.15}$$

$$+\frac{1}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\sum_{j=0}^{N}\int_{R_{N,j}(x)}dx_1\ldots dx_N B_j\left(|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\right). \tag{6.16}$$

To proceed further consider $|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2$ as a function of x_l, $(l=1,\ldots,N)$. This function vanishes at the three points $x_l=0,x_j,L$ $(j\neq l)$.

It follows that

$$\int_{R_{N,j}(x)}dx_l\,x_l\frac{\partial}{\partial x_l}|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2=-\int_{R_{N,j}(x)}dx_l|\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 \tag{6.17}$$

while

$$\int_{R_{N,j}(x)} dx_l \frac{\partial}{\partial x_l} |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 = 0. \tag{6.18}$$

Hence

$$\begin{aligned}
g_{N+2}(x,0) &= g^{(\infty)}_{N+2}(x,0) - \frac{2\rho_0}{c}(N+1)g^{(\infty)}_{N+2}(x,0) - \frac{2\rho_0}{c}\left(x\frac{\partial}{\partial x} - N\right)g^{(\infty)}_{N+2}(x,0) && (6.19)\\
&\quad + \frac{1}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\sum_{j=0}^{N} 2(j+1)\frac{\partial}{\partial x}\int_{R_{N,j}(x)} dx_1 \dots dx_N |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 && (6.20)\\
&= g^{(\infty)}_{N+2}(x,0) - \frac{2\rho_0}{c}\left(x\frac{\partial}{\partial x} + 1\right)g^{(\infty)}_{N+2}(x,0) && (6.21)\\
&\quad + \frac{2}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\frac{\partial}{\partial x}\sum_{j=0}^{N}\frac{(j+1)}{j!(N-j)!}\int_{R_{N,j}(x)} dx_1 \dots dx_N |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 && (6.22)
\end{aligned}$$

where (6.22) can be written

$$\begin{aligned}
&\frac{2}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\frac{1}{N!}\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\sum_{j=0}^{N}\xi^{j+1}\begin{pmatrix} N \\ j \end{pmatrix}\int dx_1 \dots \int dx_N |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\Big|_{\xi=1} && (6.23)\\
&= \frac{2}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\frac{1}{N!}\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\xi\prod_{l=1}^{N}\left(\int_x^L + \xi\int_0^x\right) dx_l |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\Big|_{\xi=1} && (6.24)\\
&= \frac{2}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\frac{1}{N!}\frac{\partial}{\partial x}\int_0^L dx_1 \dots dx_N |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 && (6.25)\\
&\quad + \frac{2}{c}\frac{1}{(\mathcal{N}^{(\infty)})^2}\frac{1}{N!}\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\prod_{l=1}^{N}\left(\int_x^L + \xi\int_0^x\right) dx_l |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2\Big|_{\xi=1} && (6.26)
\end{aligned}$$

we observe that

$$\frac{1}{N!}\frac{1}{(\mathcal{N}^{(\infty)})^2}\int_0^L dx_1 \dots dx_N |\det[e^{\mathrm{i}k_j\tilde{x}_l}]|^2 = g^{(\infty)}_{N+2}(x,0) = \rho_0(1-y(x)^2) \tag{6.27}$$

where

$$y(x) = \frac{\sin\left(\frac{(N+2)\pi x}{L}\right)}{(N+2)\sin\left(\frac{\pi x}{L}\right)} = \frac{1}{\rho_0}K_{N+2,L}(x,0) \tag{6.28}$$

where we note that (6.27) and (6.28) also appear in the CUE (Circular Unitary Ensemble), given by Dyson [26].
Hence (6.25) is equal to

$$\frac{2}{c}\frac{\partial}{\partial x}g^{(\infty)}_{N+2}(x,0). \tag{6.29}$$

Regarding (6.26), note that

$$\frac{\partial}{\partial \xi} \prod_{l=1}^{N} \left(\int_x^L + \xi \int_0^x \right) dx_l |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2 \Bigg|_{\xi=1} \tag{6.30}$$

$$= \frac{\partial}{\partial \xi} \left(\prod_{l=1}^{N} \int_0^L dx_l |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2 + \sum_{l=1}^{N} \int_0^x dx_l (\xi - 1) \prod_{j=1, j\neq l}^{N} \int_0^L dx_j |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2 + O((\xi-1)^2) \right) \Bigg|_{\xi=1} \tag{6.31}$$

$$= \sum_{l=1}^{N} \int_0^x dx_l \prod_{j=1, j\neq l}^{N} \int_0^L dx_j |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2 \tag{6.32}$$

$$= N \int_0^x dx_1 \int_0^L dx_2 \ldots \int_0^L dx_N |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2. \tag{6.33}$$

Therefore, (6.26) is equal to

$$\frac{2}{c} \frac{1}{(\mathcal{N}^{(\infty)})^2} \frac{1}{(N-1)!} \frac{\partial}{\partial x} \int_0^x dx_1 \int_0^L dx_2 \ldots \int_0^L dx_N |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2. \tag{6.34}$$

But for the free Fermi system the 3-point correlation function is specified by

$$\begin{aligned} g^{(\infty)}_{3,(N+2)}(x, 0, x_1) &= \frac{1}{(N-1)!} \frac{1}{(\mathcal{N}^{(\infty)})^2} \int_0^L dx_2 \ldots \int_0^L dx_N |\det[e^{\mathrm{i}k_j \tilde{x}_l}]|^2 && (6.35) \\ &= \rho_0^2 \det \begin{pmatrix} 1 & y(x) & y(x_1) \\ y(x) & 1 & y(x-x_1) \\ y(x_1) & y(x-x_1) & 1 \end{pmatrix} && (6.36) \\ &= \rho_0^2 [1 - y(x)^2 - y(x_1)^2 - y(x-x_1)^2 + 2y(x)y(x_1)y(x-x_1)] && (6.37) \end{aligned}$$

where $y(x)$ is specified by (6.28). Note that this definition is only valid for $N \geq 1$, a 3-point correlation function for a system of 2 particles does not make physical sense, $g^{(\infty)}_{3,2}(x, 0, x_1) = 0$. Now (6.34) reduces to

$$\frac{2}{c} \frac{\partial}{\partial x} \int_0^x dx_1 g^{(\infty)}_{3,(N+2)}(x, 0, x_1). \tag{6.38}$$

Finally, adding up all contributions gives the sought closed form expression (here we revert to $N + 2 \to N$)

$$g_N(x, 0) = g_N^{(\infty)}(x, 0) - \frac{2\rho_0}{c} \left(x \frac{\partial}{\partial x} + 1 \right) g_N^{(\infty)}(x, 0) + \frac{2}{c} \frac{\partial}{\partial x} g_N^{(\infty)}(x, 0) + \frac{2}{c} \frac{\partial}{\partial x} \int_0^x dx_1 g^{(\infty)}_{3,N}(x, 0, x_1) \tag{6.39}$$

correct to $\mathrm{O}(1/cL)$, valid for all $N \geq 2$. This has been checked with (5.9) and (5.15). Using (6.27) and (6.37) this can be written in the simpler form

$$\begin{aligned} g_N(x, 0) &= g_N^{(\infty)}(x, 0) + 4N \left(\frac{1}{cL} \right) \Big\{ - y(x) \left[\rho_0 y(x) + y'(x) \right] \\ &\quad + \rho_0 \frac{\partial}{\partial x} \left[y(x) \int_0^x y(x_1) y(x - x_1) dx_1 \right] \Big\} + \mathrm{O} \left(\frac{1}{cL} \right)^2 && (6.40) \end{aligned}$$

which is valid for all $N \geq 3$. We have confirmed that this result recovers (5.15). Now (6.40), and concomitantly its structure factor, are readily computed for any $N \geq 3$. The structure factor in the limit $cL \to \infty$, is given by

$$\mathcal{S}_n(N) = \begin{cases} N & \text{when } n = 0 \\ |n|/N & \text{when } 0 < |n| < N \\ 1 & \text{when } |n| \geq N. \end{cases} \tag{6.41}$$

In the thermodynamic limit, $y(x)$ becomes

$$y(x) = \frac{\sin(\rho_0 \pi x)}{\rho_0 \pi x} \tag{6.42}$$

and now the correlation function follows from (6.40) and is (where we use the appropriate scaled variable ρ_0/c)

$$\begin{aligned} g_\infty(x,0) \;&=\; \rho_0 \left(1 - \frac{\sin^2 \bar{x}}{\bar{x}^2}\right) - 4\rho_0 \Big\{ \frac{\sin \bar{x}}{\bar{x}^3} \left[\pi \bar{x} \cos \bar{x} + (\bar{x} - \pi) \sin \bar{x}\right] \\ &\quad - \frac{\partial}{\partial x} \left[y(x) \int_0^x y(x_1) y(x - x_1) dx_1 \right] \Big\} \left(\frac{\rho_0}{c}\right) + \mathrm{O}\left(\frac{\rho_0}{c}\right)^2 \end{aligned} \tag{6.43}$$

where $\bar{x} = \rho_0 \pi x$. This equation was first given by Korepin [27]. Recently in [28], using the Random Phase Approximation (RPA), which they indicated is valid to the ρ_0/c correction, the structure factor was calculated in the thermodynamic limit from which (6.43) is recovered.

Here we explicitly evaluate the integral in (6.43) with (6.42)

$$\int_0^x y(x_1) y(x - x_1) dx_1 = \frac{\sin(\bar{x}) \mathrm{Si}(2\bar{x}) + \cos(\bar{x}) [\mathrm{Ci}(2\bar{x}) - \log(2\bar{x}) - \gamma]}{\rho_0 \pi \bar{x}} \tag{6.44}$$

where γ is Euler's constant, and Si and Ci are the sine integral and cosine integral functions, respectively.

In closing this section we note that $g_N(0,0)$ in (6.40) and similarly (6.43) is zero to leading order, the free Fermi result. Deviations from zero do not begin until higher order than $1/cL$ as can be seen from these equations.

VII. CONCLUDING REMARKS

The occupation numbers for the large cL limit given in (4.21) can be continued to the asymptotically large N limit giving

$$c_n(N) \sim N^{\frac{1}{2} + \frac{\beta_n N}{cL}}. \tag{7.1}$$

Very strong evidence for the exponent β_n having the integer value 2 was presented.

In constructing the ansatz for the occupation numbers, we chose to scale cL by N from the outset. This was done because the cL large limit is now compatible with the thermodynamic limit (unlike the small cL limit as discussed after (3.20)) and further to this we found that the numerical analysis, for any finite N, in constructing the ansatz, failed without this scaling.

The question then is, how does (7.1) compare with the thermodynamic limit. In a very nice work, at a time coincident with the seminal work of the Japanese group [8, 29], the density matrix first given by Lenard in the impenetrable limit [7], [31], was extended to the $1/c$ correction [30]. They used the quantum inverse-scattering method in concert with the very important work [29] on the impenetrable Bose gas in terms of Painlevé V theory, and obtained

$$\rho(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{\frac{1}{2} - \frac{2k_\mathrm{F}}{\pi c}}}. \tag{7.2}$$

The Fourier transform of (7.2) gives the momentum distribution, with the Fermi momentum, $k_\mathrm{F} = \pi \rho_0$,

$$c(\mathbf{k}) \sim \frac{1}{|\mathbf{k}|^{\frac{1}{2} + \frac{2k_\mathrm{F}}{\pi c}}}. \tag{7.3}$$

Following the presentation given in [10], we can readily see that (7.1) is one-to-one with (7.3). An elementary way to see this immediately is to observe that $|\mathbf{k}| \sim 1/L$ and thus in the thermodynamic limit $|\mathbf{k}| \sim 1/N$. Note the integer value of 2 for the coefficient of $k_F/\pi c$ in the exponent (7.2) and (7.3).

In our work on the impenetrable Bose gas [10, 11], we studied the system in all boundary conditions, periodic, Dirichlet, Neumann, as well as for the harmonically trapped system. In all cases, we found that in the asymptotically large N limit that (7.1) held in the limit $c \to \infty$. This firmly suggests that the exponent obtained, in this large N limit, is universal, being the same for all boundary conditions and (low lying) n modes. Therefore, we anticipate the same is true now for our (7.1).

The use of periodic boundary conditions, while facilitating the mathematics, is, nonetheless, a powerful preemptor of the analytical properties of the Bose gas, a system of both continuing and immense attraction to the theoretician and the experimentalist, in concert.

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APPENDIX A: BETHE EQUATION SOLUTIONS

$N = 2$

$$k_2 = \sqrt{\frac{c}{L}}\left[1 - \frac{1}{24}(cL) + \frac{11}{5760}(cL)^2 - \frac{17}{322560}(cL)^3 - \frac{281}{154828800}(cL)^4 + \mathrm{O}(cL)^5\right] \tag{A1}$$

$N=3$

$$k_3=\sqrt{\frac{3c}{L}}\left[1-\frac{1}{24}(cL)+\frac{19}{5760}(cL)^2-\frac{299}{967680}(cL)^3+\frac{11077}{464486400}(cL)^4+\mathrm{O}(cL)^5\right] \tag{A2}$$

$N=4$

$$k_4 = \sqrt{\left(3+\sqrt{6}\right)\frac{c}{L}}\Big[1-\frac{1}{24}(cL)+\frac{31-2\sqrt{6}}{5760}(cL)^2 \tag{A3a}$$

$$+\frac{-879+86\sqrt{6}}{967680}(cL)^3+\frac{63381-5500\sqrt{6}}{464486400}(cL)^4+\mathrm{O}(cL)^5\Big] \tag{A3b}$$

$$k_3 = \sqrt{\left(3-\sqrt{6}\right)\frac{c}{L}}\Big[1-\frac{1}{24}(cL)+\frac{31+2\sqrt{6}}{5760}(cL)^2 \tag{A3c}$$

$$+\frac{-879-86\sqrt{6}}{967680}(cL)^3+\frac{63381+5500\sqrt{6}}{464486400}(cL)^4+\mathrm{O}(cL)^5\Big] \tag{A3d}$$

$N=5$

$$k_5 = \sqrt{\left(5+\sqrt{10}\right)\frac{c}{L}}\Big[1-\frac{1}{24}(cL)+\frac{39-2\sqrt{10}}{5760}(cL)^2 \tag{A4a}$$

$$+\frac{-1511+118\sqrt{10}}{967680}(cL)^3+\frac{165589-13196\sqrt{10}}{464486400}(cL)^4+\mathrm{O}(cL)^5\Big] \tag{A4b}$$

$$k_4 = \sqrt{\left(5-\sqrt{10}\right)\frac{c}{L}}\Big[1-\frac{1}{24}(cL)+\frac{39+2\sqrt{10}}{5760}(cL)^2 \tag{A4c}$$

$$+\frac{-1511-118\sqrt{10}}{967680}(cL)^3+\frac{165589+13196\sqrt{10}}{464486400}(cL)^4+\mathrm{O}(cL)^5\Big] \tag{A4d}$$

We have also computed these expansions out to $N=10$, due to the inordinate complexities of the numbers, the expansions for $N\geq 6$ are known only in decimal form.

The leading term of k_j is precisely related to the jth zero of the Nth polynomial, as mentioned in Section II. Note also the universality of the coefficient of the (cL) term, that is $-1/24$ (as in (2.8)).

APPENDIX B: FREDHOLM DETERMINANTS

Here the Fredholm minor (4.12) will be related to a multiple integral, which in turn implies the Toeplitz determinant form (4.4). Consider the multiple integral

$$\begin{aligned} A_N(x,y) &= \left(\int_0^L+\lambda\int_y^x\right)dx_2\ldots\left(\int_0^L+\lambda\int_y^x\right)dx_N \\ &\times\prod_{j=2}^N 2\sin[(x-x_j)/L]2\sin[\pi(y-x_j)/L]\prod_{2\leq j<k\leq N}\{2\sin[\pi(x_k-x_j)/L]\}^2 \end{aligned} \tag{B1}$$

Setting

$$g(u)=(1+\lambda\chi^{(u)}_{[y,x]})2\sin\left[\pi(x-u)/L\right]2\sin\left[\pi(u-y)/L\right] \tag{B2}$$

where $\chi^{(u)}_{[y,x]}=1$ for $u\in[y,x]$ and 0 otherwise, allows this to be written

$$A_N(x,y)=\int_0^L dx_2\ldots\int_0^L dx_N\prod_{l=2}^N g(x_l)\prod_{1\leq j<k\leq N}|e^{2\mathrm{i}\pi x_k/L}-e^{2\mathrm{i}\pi x_j/L}|^2. \tag{B3}$$

By a well known identity (see e.g. Szegö [24]) this is equal to a Toeplitz determinant,

$$A_N(x,y) = (N-1)!\det\left[\int_0^L du\, g(u) e^{2\mathrm{i}\pi u(j-k)/L}\right]_{j,k=1,\dots,N-1}. \tag{B4}$$

On the other hand the integral in $A_N(x,y)$ can be expanded as a power series in λ. For this define

$$\phi_N(x_1, x_2, \dots, x_N) = \prod_{1\le j<k\le N} (e^{2\mathrm{i}\pi x_k/L} - e^{2\mathrm{i}\pi x_j/L}) \tag{B5}$$

and introduce the free Fermi type distribution

$$\begin{aligned} \rho_N^{\mathrm{FF}}(x,y;x_2,\dots,x_n) \;&=\; \frac{(N-1)!}{(N-n)!C_{N,L}} \int_0^L dx_{n+1}\dots\int_0^L dx_N \\ &\times \phi_N(x,x_2,\dots,x_N)\overline{\phi(y,x_2,\dots,x_N)} \end{aligned} \tag{B6}$$

where $C_{N,L}$ is such that

$$\rho_N^{\mathrm{FF}}(x,x) = \frac{N}{L} \tag{B7}$$

and thus given by

$$NC_{N,L} = \int_0^L dx_1 \dots \int_0^L dx_N |\phi(x_1,\dots,x_N)|^2. \tag{B8}$$

Now, writing the integrand in (B1) in terms of (B5), expanding in λ, and making use of the definition (B6) we see

$$A_N(x,y) = C_{N,L}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_y^x dx_2\dots\int_y^x dx_{n+1}\rho_N^{\mathrm{FF}}(x,y;x_2,\dots,x_{n+1}). \tag{B9}$$

A straightforward calculation (see e.g. Forrester [25]) gives the determinant form

$$\rho_N^{\mathrm{FF}}(x,y;x_2,\dots,x_{n+1}) = \det\begin{bmatrix} K_{NL}(x,y) & [K_{NL}(x,u_k)]_{k=1,\dots,n} \\ [K_{NL}(u_j,y)]_{j=1,\dots,n} & [K_{NL}(u_j,u_k)]_{j,k=1,\dots,n}\end{bmatrix} \tag{B10}$$

and furthermore shows from (B8) that

$$C_{N,L} = (N-1)!L^N. \tag{B11}$$

Substituting (B11) and (B10) in (B9) and comparing with (4.12) shows

$$\frac{1}{\lambda}\Delta_1\begin{pmatrix} x \\ 0 \end{pmatrix};\lambda\Bigg) = \frac{1}{(N-1)!L^N}A_N(x,0). \tag{B12}$$

With $A_N(x,y)$ in its Toeplitz determinant form (B4), this gives (4.4).

N	$c_0^{*(0)}(N)$	$c_0^{*(1,1)}(N)$	$c_0^{*(1,2)}(N)$	$c_0^{*(1)}(N)$	$c_1^{*(0)}(N)$	$c_1^{*(1,1)}(N)$	$c_1^{*(1,2)}(N)$	$c_1^{*(1)}(N)$	$c_2^{*(0)}(N)$	$c_2^{*(1,1)}(N)$	$c_2^{*(1,2)}(N)$	$c_2^{*(1)}(N)$
2	0.810569	−0.757722	4	3.24228	0.0900633	−0.227763	−1.33333	−1.56110	0.00360253	0.219594	−0.266667	−0.0470731
3	0.702151	−1.78709	8	6.21291	0.111111	−0.924374	−1	−1.92437	0.0328356	0.249760	−1.33333	−1.08357
4	0.629414	−2.96469	11.9525	8.98781	0.116609	−1.85190	−0.111842	−1.96375	0.0464792	−0.160426	−1.40975	−1.57018
5	0.576137	−4.23863	15.8505	11.6119	0.117046	−2.92064	1.05226	−1.86838	0.0533164	−0.818135	−0.995603	−1.81374
6	0.534872	−5.58153	19.6958	14.1143	0.115561	−4.08661	2.38235	−1.70426	0.0568683	−1.63720	−0.293205	−1.93041
7	0.50165	−6.9769	23.4922	16.5153	0.113315	−5.32460	3.82394	−1.50066	0.0586798	−2.57138	0.598918	−1.97246
8	0.474130	−8.41392	27.2440	18.8301	0.110796	−6.61868	5.34600	−1.27269	0.0595055	−3.59273	1.62585	−1.96689
9	0.450832	−9.88502	30.9548	21.0698	0.108225	−7.95803	6.92912	−1.02891	0.0597461	−4.68294	2.75391	−1.92903
10	0.430766	−11.3847	34.6282	23.2435	0.105704	−9.33488	8.56031	−0.774574	0.0596284	−5.82927	3.96095	−1.86831
11	0.413239	−12.9087	38.2671	25.3583	0.103278	−10.7435	10.2304	−0.513070	0.0592871	−7.02245	5.23161	−1.79084
12	0.397753	−14.4539	41.8741	27.4202	0.100969	−12.1794	11.9327	−0.246672	0.0588057	−8.25550	6.55476	−1.70074
13	0.383935	−16.0177	45.4517	29.4340	0.0987797	−13.6392	13.6622	0.0230389	0.0582372	−9.52303	7.92208	−1.60095
14	0.371504	−17.5979	49.0018	31.4039	0.0967099	−15.1200	15.4150	0.294935	0.0576162	−10.8207	9.32717	−1.49356
15	0.360239	−19.1928	52.5264	33.3335	0.0947542	−16.6196	17.1878	0.568195	0.0569656	−12.1451	10.7650	−1.38014
16	0.349968	−20.8010	56.0269	35.2259	0.0929060	−18.1360	18.9782	0.842211	0.0563009	−13.4934	12.2315	−1.26189
17	0.34055	−22.4213	59.5049	37.0836	0.0911579	−19.6677	20.7842	1.11653	0.0556325	−14.8632	13.7235	−1.13973
18	0.331874	−24.0525	62.9616	38.9091	0.0895030	−21.2133	22.6041	1.39080	0.0549676	−16.2524	15.2380	−1.01437
19	0.323844	−25.6939	66.3983	40.7044	0.0879343	−22.7715	24.4363	1.66475	0.0543109	−17.6594	16.7730	−0.886401
20	0.316385	−27.3446	69.8160	42.4715	0.0864452	−24.3415	26.2797	1.93819	0.0536657	−19.0827	18.3264	−0.756273
21	0.309431	−29.0039	73.2157	44.2118	0.0850299	−25.9223	28.1333	2.21096	0.0530340	−20.5210	19.8966	−0.624366
22	0.302927	−30.6712	76.5983	45.9271	0.0836828	−27.5132	29.9961	2.48292	0.0524171	−21.9732	21.4822	−0.490990
23	0.296826	−32.3460	79.9645	47.6185	0.0823990	−29.1133	31.8673	2.75400	0.0518158	−23.4383	23.0819	−0.356402
24	0.291088	−34.0278	83.3151	49.2874	0.0811738	−30.7222	33.7464	3.02411	0.0512305	−24.9153	24.6945	−0.220815
25	0.285677	−35.7161	86.6509	50.9348	0.0800030	−32.3393	35.6325	3.29320	0.0506612	−26.4037	26.3193	−0.0844091
26	0.280564	−37.4106	89.9724	52.5618	0.0788828	−33.9641	37.5254	3.56123	0.0501078	−27.9026	27.9552	0.0526638
27	0.275723	−39.1110	93.2802	54.1692	0.0778098	−35.5961	39.4243	3.82817	0.0495701	−29.4114	29.6016	0.190274
28	0.271128	−40.8168	96.5749	55.7581	0.0767809	−37.2350	41.3290	4.09399	0.0490476	−30.9295	31.2578	0.328313
29	0.266761	−42.5279	99.8570	57.3292	0.0757931	−38.8803	43.2390	4.35868	0.0485400	−32.4564	32.9231	0.466684
30	0.262603	−44.2438	103.127	58.8832	0.0748437	−40.5318	45.1762	4.64439	0.0480469	−33.9918	34.5971	0.605307
31	0.258637	−45.9645	106.385	60.4207	0.0739304	−42.1891	47.0737	4.88464	0.0475677	−35.5348	36.2791	0.744267
32	0.254849	−47.6896	109.632	61.9426	0.0730510	−43.8519	48.9978	5.14590	0.0471020	−37.0856	37.9688	0.883173
33	0.251227	−49.4190	112.868	63.4493	0.0722034	−45.5201	50.9260	5.40589	0.0466492	−38.6436	39.6657	1.02209
34	0.247757	−51.1525	116.094	64.9414	0.0713858	−47.1931	52.8581	5.66498	0.0462090	−40.2085	41.3695	1.16098
35	0.244430	−52.8899	119.309	66.4194	0.0705964	−48.8710	54.7939	5.92294	0.0457808	−41.7797	43.0798	1.30008
36	0.241237	−54.6310	122.515	67.8839	0.0698336	−50.5537	56.7332	6.17953	0.0453641	−43.3572	44.7963	1.43904

Values of the large cL expansion parameters of the occupation numbers for $N = 2$ to 36.

N	$c_0^{(0)}(N)$		$c_0^{(1)}(N)$	
2	$\frac{16}{\pi^2}$	$1.62114\ldots$	$\frac{64}{\pi^2}$	$6.48456\ldots$
3	$\frac{1}{3}+\frac{35}{2\pi^2}$	$2.10645\ldots$	$8+\frac{105}{\pi^2}$	$18.6387\ldots$
4	$-\frac{2097152}{19845\pi^4}+\frac{320}{9\pi^2}$	$2.51766\ldots$	$-\frac{16777216}{19845\pi^4}+\frac{138752}{315\pi^2}$	$35.9512\ldots$
5	$\frac{1}{5}+\frac{7436429}{129600\pi^4}+\frac{4459}{216\pi^2}$	$2.88069\ldots$	$\frac{16}{3}+\frac{7436429}{12960\pi^4}+\frac{249613}{540\pi^2}$	$58.0593\ldots$
6	$\frac{19350784805830806041998131 2}{1274815781491347407812 5\pi^6}-\frac{38494793629696}{21739843125\pi^4}+\frac{4144}{75\pi^2}$	$3.20923\ldots$	$\frac{7740313922332322416799252 48}{4249385938304491359375\pi^6}-\frac{553979432056158945 28}{2589628373206875\pi^4}+\frac{85085248}{75075\pi^2}$	$84.6855\ldots$
7	$\frac{1}{7}+\frac{857606211358042978 13}{40663643328000000\pi^6}-\frac{46891706849}{317520000\pi^4}+\frac{79679}{3000\pi^2}$	$3.51155\ldots$	$\frac{24}{5}+\frac{857606211358042978 13}{2904545952000000\pi^6}+\frac{1528761661843}{2716560000\pi^4}+\frac{641803}{875\pi^2}$	$115.607\ldots$

TABLE III: Values of $c_0^{(0)}(N)$ and $c_0^{(1)}(N)$ for $N=2,3,4,5,6,7$. Note that this Table extends Table II of [10]

N	$c_1^{(0)}(N)$		$c_1^{(1)}(N)$	
2	$\frac{16}{9\pi^2}$	$0.180127\ldots$	$\frac{-832}{27\pi^2}$	$-3.12219\ldots$
3	$\frac{1}{3}$	$0.333333\ldots$	$-4-\frac{35}{2\pi^2}$	$-5.77312\ldots$
4	$\frac{-6318718976}{22325625\pi^4}+\frac{832}{25\pi^2}$	$0.466435\ldots$	$\frac{-52781507411968}{7032571875\pi^4}+\frac{1382912}{2025\pi^2}$	$-7.85498\ldots$
5	$\frac{1}{5}-\frac{18059899}{129600\pi^4}+\frac{3871}{216\pi^2}$	$0.585231\ldots$	$\frac{28}{9}-\frac{574729727}{151200\pi^4}+\frac{47201}{180\pi^2}$	$-9.34192\ldots$
6	$\frac{44585667812858633489874398740 48}{3157030290232062201551343 75\pi^6}-\frac{14163619272982528}{7456766191875\pi^4}+\frac{7984}{147\pi^2}$	$0.693364\ldots$	$\frac{317603131762611117568514042882856845312}{850617087107250872660154262734375\pi^6}-\frac{9075555026761899887820 8}{172210286818257187 5\pi^4}+\frac{8618142656}{6131125\pi^2}$	$-10.2255\ldots$

TABLE IV: Values of $c_1^{(0)}(N)$ and $c_1^{(1)}(N)$ for $N=2,3,4,5,6$

N	$c_2^{(0)}(N)$		$c_2^{(1)}(N)$	
2	$\frac{16}{225\pi^2}$	$0.00720506\ldots$	$\frac{-3136}{3375\pi^2}$	$-0.0941461\ldots$
3	$\frac{35}{36\pi^2}$	$0.0985067\ldots$	$\frac{-385}{12\pi^2}$	$-3.25072\ldots$
4	$\frac{-7408644521984}{132368630625\pi^4}+\frac{27584}{3675\pi^2}$	$0.185917\ldots$	$\frac{3710310553890062336}{458657305115625\pi^4}-\frac{6735912448}{7640325\pi^2}$	$-6.28072\ldots$
5	$\frac{1}{5}-\frac{1062347}{127008\pi^4}+\frac{325}{216\pi^2}$	$0.266582\ldots$	$-\frac{52}{9}+\frac{30358690219}{17781120\pi^4}-\frac{221909}{1080\pi^2}$	$-9.06869\ldots$
6	$\frac{1607694309681734021848756431041 3312}{821143578489359378623504509375\pi^6}-\frac{47388412779564105728}{19395048865066875\pi^4}+\frac{990928}{19845\pi^2}$	$0.34121\ldots$	$\frac{419470218911103328947555278533777045218 9184}{22124550435659595197890612373721093 75\pi^6}-\frac{2887827552035815814833635328}{1343756868042860834062 5\pi^4}+\frac{18544396881344}{8442559125\pi^2}$	$-11.5824\ldots$

TABLE V: Values of $c_2^{(0)}(N)$ and $c_2^{(1)}(N)$ for $N=2,3,4,5,6$

N	$c_0^{(1,1)}(N)$		$c_0^{(1,2)}(N)$	
2	$-8+\frac{64}{\pi^2}$	$-1.51544\ldots$	8	8
3	$-16+\frac{105}{\pi^2}$	$-5.36128\ldots$	24	24
4	$-32-\frac{16777216}{19845\pi^4}+\frac{2560}{9\pi^2}$	$-11.8587\ldots$	$32+\frac{16384}{105\pi^2}$	$47.8100\ldots$
5	$-48+\frac{7436429}{12960\pi^4}+\frac{22295}{108\pi^2}$	$-21.1931\ldots$	$\frac{160}{3}+\frac{23023}{90\pi^2}$	$79.2524\ldots$
6	$-72+\frac{774031392233232241679925248}{424938593830449135937 5\pi^6}-\frac{153979174518784}{7246614375\pi^4}+\frac{16576}{25\pi^2}$	$-33.4892\ldots$	$72-\frac{74481467421360128}{517925674641375\pi^4}+\frac{7061504}{15015\pi^2}$	$118.175\ldots$

TABLE VI: Values of $c_0^{(1,1)}(N)$ and $c_0^{(1,2)}(N)$ for $N=2,3,4,5,6$

N	$c_1^{(1,1)}(N)$		$c_1^{(1,2)}(N)$	
2	$\frac{8}{3}-\frac{832}{27\pi^2}$	$-0.455527\ldots$	$-\frac{8}{3}$	$-2.66667\ldots$
3	$-1-\frac{35}{2\pi^2}$	$-2.77312\ldots$	-3	$-3.\ldots$
4	$-\frac{352}{15}-\frac{52781507411968}{7032571875\pi^4}+\frac{65129984}{70875\pi^2}$	$-7.40762\ldots$	$\frac{352}{15}-\frac{16728064}{70875\pi^2}$	$-0.447369\ldots$
5	$-\frac{109}{3}-\frac{574729727}{151200\pi^4}+\frac{64757}{108\pi^2}$	$-14.6032\ldots$	$\frac{355}{9}-\frac{91091}{270\pi^2}$	$5.26130\ldots$
6	$-\frac{2232}{35}+\frac{3176031317626111175685140428828568 45312}{850617087107250872660154262734375\pi^6}-\frac{403734185313387287674 88}{803648005151866875\pi^4}+\frac{30246667072}{18393375\pi^2}$	$-24.5196\ldots$	$\frac{2232}{35}-\frac{269887035484109642137 6}{1095883643388909375\pi^4}-\frac{399294464}{1672125\pi^2}$	$14.2941\ldots$

TABLE VII: Values of $c_1^{(1,1)}(N)$ and $c_1^{(1,2)}(N)$ for $N=2,3,4,5,6$

N	$c_2^{(1,1)}(N)$		$c_2^{(1,2)}(N)$	
2	$\frac{8}{15}-\frac{3136}{3375\pi^2}$	$0.439187\ldots$	$-\frac{8}{15}$	$-0.533333\ldots$
3	$4-\frac{385}{12\pi^2}$	$0.749279\ldots$	-4	-4
4	$\frac{2272}{105}+\frac{3710310553890062336}{458657305115625\pi^4}-\frac{39711774208}{38201625\pi^2}$	$-0.641704\ldots$	$-\frac{2272}{105}+\frac{6032211968}{38201625\pi^2}$	$-5.63902\ldots$
5	$\frac{31}{3}+\frac{30358690219}{17781120\pi^4}-\frac{238405}{756\pi^2}$	$-4.09068\ldots$	$-\frac{145}{9}+\frac{830687}{7560\pi^2}$	$-4.97802\ldots$
6	$-\frac{3592}{105}+\frac{41947021891110332894755527853377704521891 84}{2212455043565959519789061237372109375\pi^6}-\frac{98659506842974376305885184}{418057692280001148375\pi^4}+\frac{13193444962112}{2814186375\pi^2}$	$-9.82321\ldots$	$\frac{3592}{105}+\frac{19835961754185239649887191 04}{940629807630002583843 75\pi^4}-\frac{21035938004992}{8442559125\pi^2}$	$-1.75923\ldots$

TABLE VIII: Values of $c_2^{(1,1)}(N)$ and $c_2^{(1,2)}(N)$ for $N=2,3,4,5,6$